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Exploiting work of Wiles, DeligneRibet, FröhlichTaylor, and Serre for equivariant Iwasawa theory

This talk concerns an equivariant extension of classical Iwasawa theory. Here is the set-up.

> k_1/k is a finite Galois extension of totally real number fields, G_1 its Galois group, l an odd prime number, and S a finite set of places of k containing all archimedean places and those dividing l [1]. Moreover, k_∞ is the cyclotomic $\mathbb{Z}_l$-extension of k, $\Gamma_k = \mathrm{Gal}(k_\infty/k) = \langle\gamma_k\rangle \simeq \mathbb{Z}_l$, $K = k_1k_\infty$, $G = \mathrm{Gal}(K/k)$, $H = \mathrm{Gal}(K/k_\infty)$ (hence H is a finite normal subgroup of G). And finally, M is the maximal abelian l-extension of K which is unramified outside S, so the Galois group $X = \mathrm{Gal}(M/K)$ is the Iwasawa module. One knows that X is a finitely generated *torsion* ΛG-module with $\Lambda G = \mathbb{Z}_l[[G]]$ denoting the completed group ring of G over $\mathbb{Z}_l$.

If G_1, and thus G, is abelian, and if $l \nmid |G_1|$ ($l \nmid |H|$ would suffice), then ΛG splits into the direct sum $\bigoplus_\chi \mathbb{Z}_l[\chi][[T]]$ of power series rings as shown, with the sum ranging over the irreducible characters of G_1 (respectively of H) modulo the $\mathrm{Gal}(\mathbb{Q}_l{}^c/\mathbb{Q}_l)$-action, and, accordingly, X splits into the direct sum of the χ-eigenspaces X^χ whose characteristic polynomials (in $\mathbb{Z}_l[\chi][[T]]$) can be obtained from the S-truncated Artin L-functions $L_{l,k_1/k,S}(s,\chi)$. Indeed, Iwasawa for $k = \mathbb{Q}$ and Pierrette Cassou-Noguès (also Barsky and DeligneRibet) for arbitrary totally real k [2] have shown

$$L_{l,k_1/k,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)} \quad (s \in \mathbb{Z}_l\,,\ u \in 1 + l\mathbb{Z}_l \text{ satisfying } \zeta_{l^\infty}^{\gamma_k} = \zeta_{l^\infty}^{u})$$

with $G_{\chi,S}(T) \in \mathbb{Z}_l[\chi][[T]]$ and $H_\chi(T) \in \mathbb{Z}_l[\chi][T]$ an easy polynomial reflecting the possible pole at $s = 1$ of $L_{l,k_1/k,S}(s,\chi)$. The Main Conjecture of Iwasawa theory, proved by MazurWiles 1984 for $k = \mathbb{Q}$ and by Wiles 1990 for any totally real k, states that, up to a unit in $\mathbb{Z}_l[\chi][[T]]$, $G_{\chi,S}(T)$ is the characteristic polynomial of X^χ.

Our (and everything in this talk is taken from joint work with A. Weiss) - our motivation for searching an equivariant extension of this Main Conjecture, that is to say, an Iwasawa theory applying to Galois extensions k_1/k without assuming that G_1 is abelian or $l \nmid |G_1|$, arose from our work on Chinburg's conjecture which relates the root numbers in the functional equation of the Artin L-functions with the Galois structure of the unit group of k_1. In 2002 we proved this conjecture away from 2 for absolutely abelian totally real k_1 and felt that in order to study more general cases we needed an equivariant Iwasawa theory.

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[1] No result in this talk depends on a special choice of S.

[2] $l \nmid |G_1|$ is not needed here

The first problem that comes up in the general situation K/k is how to get some control on the ΛG-module X. Encouraged by the FröhlichTaylor work on the Galois structure of the ring $\mathfrak{o}_N$ of integers of N for tame Galois number field extensions N/N_0, we lifted their K-theoretical set-up to the Iwasawa setting and looked for an element $\mho$ in the Grothendieck group $K_0T(\Lambda G)$ of finitely generated torsion ΛG-modules of finite projective dimension, which would reflect the Iwasawa module X. Note that X is torsion but generally does not have finite projective dimension. Here is our refinement $\mho \in K_0T(\Lambda G)$ of X :

The exact sequence of Galois groups, $X \rightarrowtail \mathrm{Gal}(M/k) \twoheadrightarrow G$, induces the exact sequence

$$\Delta(\mathrm{Gal}(M/k), X) \rightarrowtail \Lambda(\mathrm{Gal}(M/k)) \twoheadrightarrow \Lambda G$$

of group rings, which, on dividing out $\Delta(\mathrm{Gal}(M/k), X)\Delta(X)$, yields an exact ΛG-module sequence $X \rightarrowtail Y \twoheadrightarrow \Delta(G)$, since $\Delta(\mathrm{Gal}(M/k), X)/\Delta(\mathrm{Gal}(M/k), X)\Delta(X) \simeq X$, as X is abelian. The Šafarevič Weil theorem giving the extension class of the above Galois group sequence implies that Y has finite projective dimension. Now it is possible to fill in the shown vertical maps in the commutative diagram

$$\begin{array}{ccccccc} & & \Lambda G & = & \Lambda G & & \\ & & \Psi \downarrow & & \psi \downarrow & & \\ X & \rightarrowtail & Y & \twoheadrightarrow & \Lambda G & \twoheadrightarrow & \mathbb{Z}_l \\ \| & & \downarrow & & \downarrow & & \| \\ X & \rightarrowtail & \mathrm{coker}\,\Psi & \to & \mathrm{coker}\,\psi & \twoheadrightarrow & \mathbb{Z}_l \end{array}$$

from which one arrives at the torsion modules $\mathrm{coker}\,\psi, \mathrm{coker}\,\Psi$ which obviously have finite projective dimension.

Set $\mho = [\mathrm{coker}\,\Psi] - [\mathrm{coker}\,\psi] \in K_0T(\Lambda G)$; $\mho$ is independent of special choices of Ψ or ψ. If G was abelian and $l \nmid |H|$, then the bottom sequence in the diagram could just be read as $[X] - [\mathbb{Z}_l] = \mho$, and the classical Main Conjecture says that $[X]$ is represented by $G_{\chi,S}(T)$ and $[\mathbb{Z}_l]$ by the 'pole' $H_\chi(T)$.

In order to find a representing homomorphism of $\mho$ in general, we invoke the localization sequence of K-theory to the pair

$$\Lambda G \quad \text{and} \quad \mathcal{Q}G = \{\, \tfrac{a}{b}, a \in \Lambda G, 0 \neq b \in \Lambda\Gamma \text{ for some central } \Gamma \simeq \mathbb{Z}_l \text{ in } G\}$$ [3],

which displays the map ∂ in the exact sequence $K_1(\Lambda G) \to K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0T(\Lambda G)$. Moreover, the reduced norm on the Wedderburn components of the semi-simple $\mathcal{Q}\Gamma_k$-algebra $\mathcal{Q}G$ yields the map

$$\mathrm{Det} \,:\, K_1(\mathcal{Q}G) \to \mathrm{Hom}^*(R_l(G), (\mathcal{Q}^{\mathrm{c}}\Gamma_k)^\times)\,,$$

where $R_l(G)$ is the $\mathbb{Z}$-span of the ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible characters χ of G with *open* kernel, $\mathcal{Q}^{\mathrm{c}}\Gamma_k = {\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_k$, and $*$ indicates a natural compatibility of the homomorphisms with, on the one hand, the action of $\mathrm{Gal}({\mathbb{Q}_l}^{\mathrm{c}}/\mathbb{Q}_l)$ and, on the other, twisting the argument χ with a W-type character ρ, i.e., a ρ that is inflated from $G/H \simeq \Gamma_k$.

[3] $\mathcal{Q}G$ does not depend on the choice of Γ.

Here is the definition of 'Det' : Note first that the Wedderburn components W_χ of $\mathcal{Q}G$ are in 1-1 correspondence with the χ modulo the $*$-action. Now, $K_1(\mathcal{Q}G) \approx (\mathcal{Q}G)^\times/[(\mathcal{Q}G)^\times, (\mathcal{Q}G)^\times]$, hence represent $\varepsilon \in K_1(\mathcal{Q}G)$ by $e \in (\mathcal{Q}G)^\times$ and set $\mathrm{Det}\,(\varepsilon) = [\chi \mapsto \mathrm{nr}_{W_\chi}(e_\chi)]$ with e_χ the component of e in W_χ. So $\mathrm{nr}_{W_\chi}(e_\chi)$ is in the centre of W_χ, which is naturally contained in $\mathcal{Q}^c\Gamma_k$.

Putting things together, we have arrived at the triangle

$$\begin{array}{ccc} K_1(\mathcal{Q}G) & \xrightarrow{\partial} & K_0T(\Lambda G) \\ \mathrm{Det}\downarrow & & \\ \mathrm{Hom}^*(R_l(G), (\mathcal{Q}^c\Gamma_k)^\times) & & \end{array}$$

with the distinguished Iwasawa element $\mho$ in $K_0T(\Lambda G)$.

Define the *Iwasawa L-function* L by

$$L(\chi) = \frac{G_{\chi,S}(\gamma_k - 1)}{H_\chi(\gamma_k - 1)} \in \mathcal{Q}^c\Gamma_k\,;$$

this is independent of a special choice of γ_k. The standard properties of the Artin L-functions imply $L \in \mathrm{Hom}^*(R_l(G), (\mathcal{Q}^c\Gamma_k)^\times)$. Note here that the characters χ have open kernel, so χ is actually a character of the Galois group of a finite totally real Galois extension of k, for which the representation $G_{\chi,S}(u^s-1)/H_\chi(u^s-1)$ of the S-truncated l-adic Artin L-function has been shown by Greenberg on using Brauer induction.

The 'equivariant main conjecture' of Iwasawa theory, which we formulated in 2002/2003, relates this Iwasawa L-function and $\mho$ by means of a 'Stickelberger element' Θ in $K_1(\mathcal{Q}G)$ like so

(emc) $\qquad \exists\ \Theta \in K_1(\mathcal{Q}G):\ \mathrm{Det}\,(\Theta) = L \text{ and } \partial(\Theta) = \mho\,.$

In the classical case this is exactly the Main Conjecture. If $SK_1(\mathcal{Q}G)$ was trivial, as would follow from a conjecture of Suslin concerning center fields of cohomological dimension 3 ($\mathcal{Q}\Gamma_k$ is such a field), we could actually say that *the Iwasawa L-function L determines the Iwasawa module* X (even its refinement $\mho$); (emc) exhibits a bridge between L and $\mho$ by means of Θ, though.

By the way, the corresponding bridge in the work of FröhlichTaylor connects the Cassou-Noguès root number class in the Hom-group with the difference $[\mathfrak{o}_N] - [\bigoplus_\sigma \mathfrak{o}_{N_0} a^\sigma]$ in $K_0T(\mathbb{Z}[\mathrm{Gal}(N/N_0)])$, where $\{a^\sigma, \sigma \in \mathrm{Gal}(N/N_0)\}$ is a normal basis of N over N_0.

We can prove (emc) only when the Iwasawa invariant $\mu_{K/k}$ vanishes, i.e., when the l-torsion part of X is finite (as Iwasawa conjectures). For simplicity, from now on I will also assume that K/k is a pro-l extension.

Denote by $\Lambda_\bullet G = \{\frac{a}{b} : a \in \Lambda G,\ 0 \neq b \in \Lambda\Gamma$ not divisible by $l\,; \Gamma \simeq \mathbb{Z}_l$ central in $G\}$ the localization of ΛG at l [4] and define analogously $X_\bullet, \mho_\bullet, \partial_\bullet$. Observe that ∂, for pro-l groups G, maps $K_1(\mathcal{Q}G)$ surjectively on $K_0T(\Lambda G)$. Note also that $\mu_{K/k} = 0$ implies $X_\bullet = \mho_\bullet = 0$.

As $\mho \in \mathrm{im}\,(\partial)$, there is a preimage x of $\mho$ in $K_1(\mathcal{Q}G)$ which comes from $K_1(\Lambda_\bullet G)$. Now, if we knew that $L = \mathrm{Det}\,\lambda$ for some $\lambda \in K_1(\Lambda_\bullet G)$, the element $\mathrm{Det}\,(x\lambda^{-1})$, by classical Iwasawa

[4] Again, $\Lambda_\bullet G$ does not depend on the choice of Γ.

theory, would belong to $\mathrm{Hom}^*(R_l(G),(\Lambda^c\Gamma_k)^\times)$ and at the same time to $\mathrm{Det}\,K_1(\Lambda_\bullet G)$, hence already to $\mathrm{Det}\,K_1(\Lambda G)$ as follows from extending the FröhlichTaylor integral logarithm to the Iwasawa setting; in fact, by means of this logarithm determinants can be detected. Thus, $\mathrm{Det}\,(x\lambda^{-1}) = \mathrm{Det}\,(z)$ with $z \in K_1(\Lambda G)$, asnd consequently $L = \mathrm{Det}\,(xz^{-1})$, $\partial(xz^{-1}) = \mho$. This yields

Theorem A. $\exists\,\Theta \iff L = \mathrm{Det}\,(\lambda) \in \mathrm{Det}\ K_1(\Lambda_\bullet G)$

Such $\lambda \in K_1(\Lambda_\bullet G)$ will be called *pseudomeasures.* For abelian G the pseudomeasure is uniquely determined by L and coincides with Serre's pseudomeasure introduced in his 1978 paper on the work of DeligneRibet.

The integral logarithm $\mathbb{L}$ in the Iwasawa setting is defined by the commutative square

$$\begin{array}{ccc} K_1(\Lambda_\bullet G) & \stackrel{\mathbb{L}}{\to} & T(\mathcal{Q}G) \\ \mathrm{Det}\downarrow & & \mathrm{Tr}\downarrow\simeq \\ \mathrm{HOM}(R_l(G),(\Lambda_\bullet^c\Gamma_k)^\times) & \stackrel{\mathbf{L}}{\to} & \mathrm{Hom}^*(R_l(G),\mathcal{Q}^c\Gamma_k) \end{array}$$

with $T(\mathcal{Q}G) \stackrel{\mathrm{def}}{=} \mathcal{Q}G/\langle ab-ba\rangle$, 'Tr' the composition of the reduced traces on the Wedderburn components of $\mathcal{Q}G$, so 'Tr' is an isomorphism, and with

$$(\mathbf{L}(f))(\chi) = \frac{1}{l}\log\frac{f(\chi)^l}{\Psi(f(\psi_l\chi))}$$

for $f \in \mathrm{HOM}$, the subgroup of the $f \in \mathrm{Hom}^*$ satisfying $f(\chi) \in (\Lambda_\bullet^c\Gamma_k)^\times \stackrel{\mathrm{def}}{=} (\mathbb{Z}_l{}^c \otimes_{\mathbb{Z}_l} \Lambda_\bullet\Gamma_k)^\times$ and the congruence

$$f(\chi)^l \equiv \Psi(f(\psi_l\chi) \mod l$$

where Ψ is induced by the endomorphism of $\mathbb{Z}_l[[T]]$ sending T to $(1+T)^l-1$ and where ψ_l is the Adams operation $\psi_l\chi(g) = \chi(g^l)$ for $g \in G$. As a matter of fact, all $f \in \mathrm{Det}\ K_1(\Lambda_\bullet G)$ belong to HOM and also $L \in \mathrm{HOM}$ as follows from the Main Conjecture in the abelian situation and, in general, from applying explicit Brauer induction (due to Snaith and Boltje). [5]

The above diagram allows us to define the *logarithmic pseudomeasure* $t \in T(\mathcal{Q}G)$ by

$$\mathrm{Tr}(t) = \mathbf{L}(L)\,;$$

however, we do not know whether $t \in \mathrm{im}\,(\mathbb{L})$. And, in fact,

Theorem B. $t = \mathbb{L}(y)$ *with* $y \in K_1(\Lambda_\bullet G) \implies \mathrm{Det}\,(y)$ *differs from* L *by a torsion element in* HOM, *which is trivial precisely when, for all sections* G' *of* G *(i.e.,* $G' = U/N$ *with* U *open in* G *and* $N \lhd U$ *finite) having an abelian subgroup* A' *of index* l*, the*

$$\text{torsion congruence}\ \ \mathrm{ver}^{A'}_{(G')^{ab}}(\lambda_{(G')^{ab}}) \equiv \lambda_{A'} \mod \mathrm{tr}_{G'/A}\Lambda_\bullet A'\ \ \textit{holds,}$$

[5] Since $\Lambda_\bullet G$ is not l-complete, the use of 'log' may require to replace $\Lambda_\bullet G$ by its l-completion; this does not affect our arguments though.

in which $\lambda_{(G')^{ab}}, \lambda_{A'}$ *are the pseudomeasures attached to the abelian groups* $(G')^{ab}, A'$, *respectively, and where* $\mathrm{ver}^{A'}_{(G')^{ab}}$ *is the group transfer* $(G')^{ab} \to A'$ *lifted to* $K_1(\Lambda_\bullet(G')^{ab}) \to K_1(\Lambda_\bullet A')$.

Necessity of the torsion congruence is obtained from fine-tuning C.T.C. Wall's work on torsion determinants of group rings of finite groups of prime power order, which yields $\mathrm{ver}^{A'}_{(G')^{\mathrm{ab}}}\mathrm{defl}^{(G')^{\mathrm{ab}}}_{G'}\varepsilon \equiv \mathrm{res}\,^{A'}_{G'}\varepsilon \mod \mathrm{tr}\,_{G'/A'}(\Lambda_\bullet A)$ for units $\varepsilon \in \Lambda_\bullet A$ (a fact that has already been used in the proof of Theorem A). It then takes the DeligneRibet q-expansion principle for Hilbert modular forms to confirm the last congruence with $\mathrm{defl}^{(G')^{\mathrm{ab}}}_{G'}\varepsilon$, $\mathrm{res}\,^{A'}_{G'}\varepsilon$ replaced by $\lambda_{(G')^{\mathrm{ab}}}$, $\lambda_{A'}$, respectively.

Remark. FröhlichTaylor have a somewhat analogous result. However, with them it is the DavenportHasse formula which takes care of the torsion elements in Hom; on the other hand, the 'equivalent' of '$t \in \mathrm{im}\,\mathbb{L}$' is a direct consequence of the tameness assumption.

To finish the proof of (emc), i.e., $t \in \mathrm{im}\,\mathbb{L}$, we, however, need three more ingredients.

Firstly, a restriction map $\mathrm{Res}^U_G : T(\mathcal{Q}G) \to T(\mathcal{Q}U)$ for open subgroups U of G, which commutes with the obvious restriction $\mathrm{res}\,^U_G : K_1(\Lambda_\bullet G) \to K_1(\Lambda_\bullet U)$. This 'Res', which is a complicated map and differs quite drastically from the canonical 'res' on $T(\mathcal{Q}G)$, together with the natural deflation maps induced by $U \twoheadrightarrow U/N$ to finite normal subgroups N of U, allows us to pass to sub- and factor groups and thus to employ an induction argument on the order of $G/Z(G)$, which is simultaneously carried out on both sides of the logarithmic diagram: on the left (multiplicative) side by using further congruences on abelian pseudomeasures, and on the right (additive) side where we profit from the uniqueness of t. I should perhaps remark that there is no analogue of 'Res' in the work of FröhlichTaylor.

Secondly, we need a new congruence on units in $\Lambda_\bullet G$, namely the so-called *Möbius-Wall congruence*

Theorem C1. $\sum_{A\le U\le G}\mu_Q(U/A)\mathrm{ver}^A_U(\mathrm{res}\,^U_G\varepsilon) \equiv 0 \mod \mathrm{tr}\,_Q(\Lambda_\bullet A)$ *for units* $\varepsilon \in \Lambda_\bullet G$.

Here, A is an abelian normal open subgroup of G, $Q \overset{\mathrm{def}}{=} G/A$ and μ_Q is the Möbius function on the partially ordered set of subgroups of the finite l-group Q, so $\mu_Q(1) = 1$, $\mu_Q(Q') = -\sum_{1\le Q''<Q'}\mu_Q(Q'')$ for $1 \ne Q' \le Q$.

And thirdly, this congruence between Serre's abelian pseudomeasures

Theorem C2. $\sum_{A\le U\le G}\mu_Q(U/A)\mathrm{ver}^A_U(\lambda_{U^{ab}}) \equiv 0 \mod \mathrm{tr}\,_Q(\Lambda_\bullet A)$

So $\mathrm{res}\,^U_G\varepsilon$ has been replaced by $\lambda_{U^{\mathrm{ab}}} \in K_1(\Lambda_\bullet U^{\mathrm{ab}})$ in Theorem C1. Note that C2 repeats the torsion congruence if $[G : A] = l$.

The proof of Theorem C1 is very technical and, to be honest, ugly. Theorem C2 can again be derived from the DeligneRibet q-expansion principle for Hilbert modular forms (and thus their paper, besides the one of Wiles on the Main Conjecture, is the second basic ingredient in our proof of (emc)). Both congruences combined permit us to run the above mentioned induction argument (on the index $[G : Z(G)]$) for getting '$t \in \mathrm{im}\,(\mathbb{L})$'. However, carrying out all this is probably too much for now and, already for time restrictions, I better don't go closer into the details of the proof anymore.